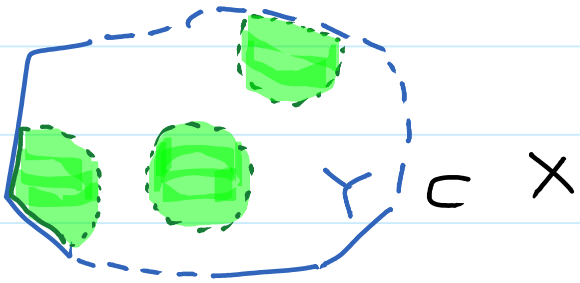
Lect13-0303-Finite Product

Friday, March 4, 2016 11:35 AM

Subspace Given $(X,\mathcal{T})$ and $Y \subset X$.

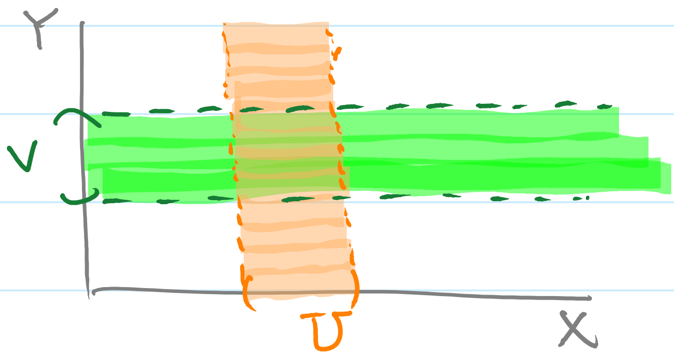
$\mathcal{T}|_Y = \{ G \cap Y : G \in \mathcal{T} \}$ is a topology of Y, called

induced or relative or subspace topology



Finite Product. Given $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$, the product topology $\mathcal{T}_{X \times Y}$ of $X \times Y$ is generated

by $\mathcal{S} = \{ X \times V : V \in \mathcal{T}_Y \} \cup \{ U \times Y : U \in \mathcal{T}_X \}$



Take finite intersections, we have a base

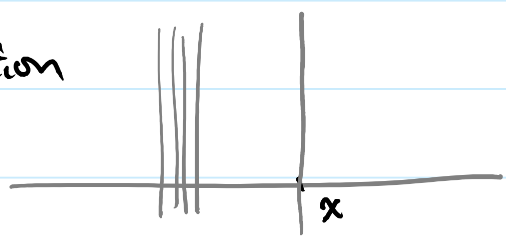
$\mathcal{B} = \{ U \times V : U \in \mathcal{T}_X , V \in \mathcal{T}_Y \}$

It contains sets of the form "rectangles".

Examples

(a) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
$\cup$
x

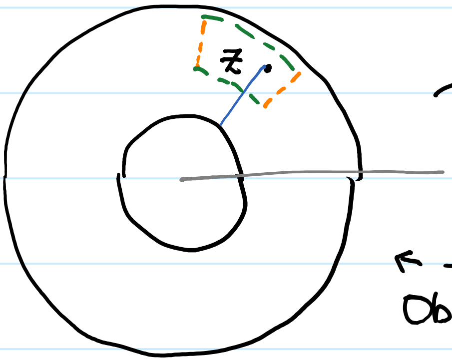
Intuition



Example (Annulus, Cylinder)

$A = \{ z \in \mathbb{C} : 1 \le |z| \le 2 \} \subset \mathbb{C} = \mathbb{R}^2$

is given the subspace topology from $\mathbb{R}^2$.



$$z \xrightarrow{\varphi} \left(\frac{z}{|z|}, |z| - 1 \right) \in S^1 \times [0,1]$$

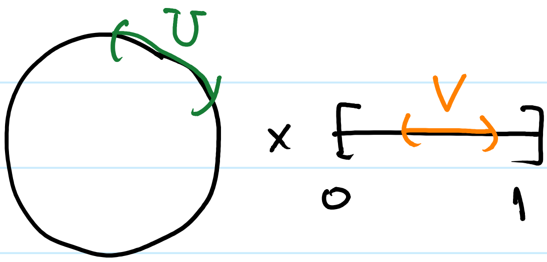
Obviously, φ^{-1} exists

Mathematically,

$S^1 \subset \mathbb{R}^2$ as subspace

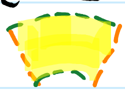
$[0,1] \subset \mathbb{R}$ as subspace

$S^1 \times [0,1]$ product space



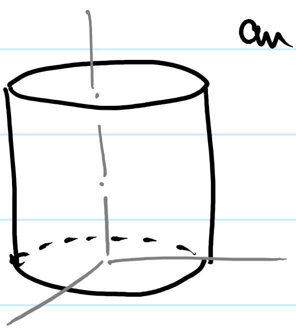
Basic open set of $S^1 \times [0,1]$ is $U \times V$

$U \in \mathcal{T}_{S^1}$ and $V \in \mathcal{T}_{[0,1]}$

Then $\varphi^{-1}(U \times V) =$  $\subset \mathbb{R}^2$ is open

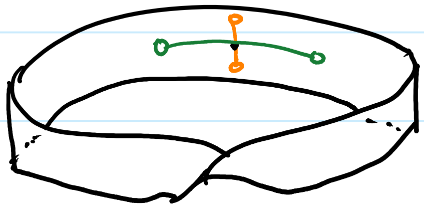
Also, every open set in $\mathbb{R}^2$ is a union of such shape, $\therefore$ φ is an open mapping and hence a homeomorphism.

Exercise. Intuitively, $S^1 \times [0,1]$ is formed by putting an $[0,1]$ as every point of S^1.



A cylinder $\subset \mathbb{R}^3$ as subspace is homeomorphic to $S^1 \times [0,1]$.

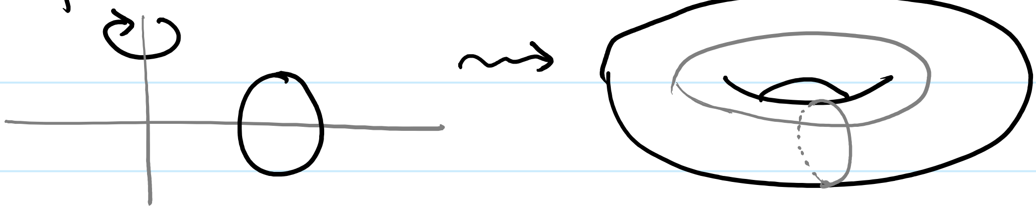
Non-example. In the above, a main step is to show every point has basic open set of the form $U \times V$ for $U \in S^1$ and $V \in [0,1]$.
But, this does not guarantee a product. There must be a bijection at first.



The Möbius strip has basic open sets of the form $U \times V$ for $U \in \mathcal{J}_{S^1}$, $V \in \mathcal{J}_{[0,1]}$.

Sphere. $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$
It clearly has a subspace topology from $\mathbb{R}^3$.
Its also has basic open sets of the form $U \times V$ where $U \in \mathcal{J}_{\mathbb{R}}$ and $V \in \mathcal{J}_{\mathbb{R}}$ given by the spherical coordinates.

Torus. $\mathbb{T} \subset \mathbb{R}^3$ can be formed as a surface of revolution



Exercise. Show that $\mathbb{T}$ is homeomorphic to the product space $S^1 \times S^1$.